

ON THE CLASSIFICATION OF SURFACES OF GENERAL TYPE WITH NON-BIRATIONAL BICANONICAL MAP AND DU VAL DOUBLE PLANES

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This paper is dedicated to the memory of prof. Paolo Francia.

ABSTRACT. We classify surfaces of general type whose bicanonical map φ_{2K} is composed with a rational map of degree 2 onto a rational or ruled surface. In particular, this is always the case if $q = 0, p_g \geq 2$ and φ_{2K} is not birational.

We prove that such a surface S either has a genus 2 pencil or is the smooth model of a double plane branched along a reduced curve with certain singularities, a configuration already suggested by Du Val in the 1950's.

In the last case we show that S has a rational pencil $|C|$ such that the general member is a smooth hyperelliptic curve of genus 3, unless K_S is ample and either $p_g(S) = 6, K_S^2 = 8$ or $p_g(S) = 3, K_S^2 = 2$.

Let S be a smooth minimal algebraic surface over the complex numbers with geometric genus $p_g(S) = h^0(S, \mathcal{O}_S(K_S))$ and irregularity $q(S) = h^1(S, \mathcal{O}_S)$. Assume that S is of general type, then the bicanonical map of S is the rational map

$$\varphi_{2K} : S \dashrightarrow S_2 \subseteq \mathbb{P}^{K_S^2 + p_g(S) - q(S)}$$

defined by the linear system $|2K_S|$, where K_S is a canonical divisor for S and S_2 is the bicanonical image.

A theorem of Xiao [17] says that S_2 is a surface unless $p_g(S) = q(S) = 0$ and $K_S^2 = 1$. On the other hand there is a *standard case* for the non birationality of φ_{2K} , that is if S has a pencil $|C|$ such that the general element $C \in |C|$ is a curve of genus 2.

By [16] if φ_{2K} is not birational and S does not present the standard case then $K_S^2 \leq 9$, thus there are finitely many families of such surfaces and it is natural to study and try to classify them.

In the 1950's Du Val suggested that examples of minimal surfaces of general type with non birational bicanonical map can be obtained in the following way.

Let X be a smooth surface and $G \subset X$ a reduced curve such that

- B) either $X = \mathbb{F}_2$ and $G = C_0 + G'$, where $G' \in |7C_0 + 14\Gamma|$ and G' has at most non essential singularities;
- D) or $X = \mathbb{P}^2$ and G is a smooth curve of degree 8;
- \mathcal{D}_n) or $X = \mathbb{P}^2$ and $G = G' + L_1 + \dots + L_n$, with $n \in \{0, 1, \dots, 6\}$ ($G = G'$ if $n = 0$), where L_1, \dots, L_n are distinct lines meeting at a point γ and G' is a curve of degree $10 + n$. The singularities of G , besides the non essential ones, are a $(2n+2)$ -tuple point at γ , a $[5, 5]$ -point lying on L_i , $i = 1, \dots, n$, possibly some 4-tuple points or $[3, 3]$ -points;

This work was partially supported by EU Research Training Network EAGER (HPRN-CT-2000-00099).

then S is the smooth minimal model of the double cover $X' \rightarrow X$ branched along G . Here \mathbb{F}_2 is the Hirzebruch surface $\mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ and Γ, C_0 its fibre and negative section with $C_0^2 = -2$.

We will refer to such examples as the *Du Val examples*, whilst by abuse of notation we will say that X' is a *Du Val double plane* (of type \mathcal{B}, \mathcal{D} or \mathcal{D}_n respectively)

Under the hypotheses $h^1(S, \mathcal{O}_S) = 0, p_g(S) \geq 3$ and that the general canonical curve is irreducible Du Val proved that if φ_{2K} is not birational and S does not present the standard case then S is one of the above examples.

More recently C.Ciliberto, P.Francia and M.Mendes Lopes have considered the same problem in [5] and [6] removing the hypothesis concerning the general canonical curve and the regularity of S . They worked it out with modern arguments and essentially they confirmed the classification of Du Val for the regular case (i.e. $q(S) = 0$).

In my PhD thesis (cfr. [2]) I proved an analogous result for regular surfaces with $p_g(S) = 2$ under the assumption that the canonical system has no fixed part.

In this article we extend the above results rephrasing Du Val's claim. For this we remark that if $q(S) = 0$ and $p_g(S) \geq 2$ then φ_{2K} is either birational or a (generically finite) morphism of degree 2 onto a rational surface.

In fact, φ_{2K} has no base points by [8] and writing $|K_S| = |M| + F$ where $|M|$ is the movable part we have that the general curve $M \in |M|$ is irreducible and $|2K_S|$ separates different curves of $|M|$. Therefore, looking at the exact sequence

$$H^0(S, \mathcal{O}_S(K_S + M)) \rightarrow H^0(M, \mathcal{O}_M(K_M)) \rightarrow 0$$

we get that if φ_{2K} is not birational the rational map $\varphi_{|K_S+M|}$ defined by the linear system $|K_S+M| \subset |2K_S|$ is not birational on a general M . Hence M is hyperelliptic and $\varphi_{2K} : S \rightarrow S_2$ is a generically finite morphism of degree 2. Therefore, S_2 is a surface of degree $2K_S^2$ in \mathbb{P}^N where $N = K_S^2 + p_g(S)$ and as $2K_S^2 < 2N - 2$, S_2 is a ruled surface. Whence, S_2 is rational since S is regular.

More generally, we may consider minimal surfaces of general type for which the bicanonical map factors through a rational map ϕ of degree 2 onto a rational or ruled surface, that is if there exists a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi_{2K}} & S_2 \\ \phi \downarrow & \searrow \phi_2 & \\ \Sigma & & \end{array}$$

where ϕ is a (generically finite) rational map of degree two and Σ is a rational or ruled surface.

Our main result is the following

Theorem 0.1. *Let S be a smooth minimal surface of general type which does not present the standard case. Then the following three conditions are equivalent:*

- a) *the bicanonical map of S factors through a rational map of degree 2 onto a rational or ruled surface*
- b) *the bicanonical map of S factors through a rational map of degree 2 onto a rational surface*
- c) *S is the smooth minimal model of a Du Val double plane.*

Moreover, let S be as in (c) (resp. (a) or (b)) then:

- d) $q(S) = 0$ unless $p_g(S) = q(S) = 1$;
- e) unless K_S is ample and $p_g(S) = 6, K_S^2 = 8$ or $p_g(S) = 3, K_S^2 = 2$, there is a rational pencil whose general member is a smooth hyperelliptic curve of genus 3 such that the bicanonical map of S induces the hyperelliptic involution on it.

We would like to remark that we get the classification of regular surfaces with $p_g(S) \geq 2$ and non birational bicanonical map. In fact, by the above remark and Theorem 0.1 it follows that:

Theorem 0.2. *Let S be a smooth minimal surface of general type with $q(S)=0$, $p_g(S) \geq 2$. Assume that the bicanonical map of S is not birational.*

Then if S does not present the standard case it is the smooth minimal model of a Du Val double plane.

We remark that Theorem 0.1 also completes the classification of regular surfaces of general type with $p_g(S) = 1$ and non birational bicanonical map.

In fact, in this case if φ_{2K} has degree 2 then S_2 is a surface of degree $2N - 2$ in \mathbb{P}^N and so it is either ruled or a $K3$. The $K3$ case is classified by D.Morrison ([10]). Otherwise, φ_{2K} has degree greater than 2 and then $K_S^2 \leq 2$ (cfr. [18]), such surfaces are classified by F.Catanese ([3]) for $K_S^2 = 1$ and by F.Catanese, O.Debarre ([4]) for $K_S^2 = 2$.

The paper is organized as follows. In section 1 we fix some notation and we recall some general facts concerning the surfaces under consideration. In §2 we work out a first easy case, then we prove a result which suffices to get $(b) \Rightarrow (c)$ of Theorem 0.1 and starting from it we prove the implication $(b) \Rightarrow (c)$ in §3. In §4 we prove $(c) \Rightarrow (b)$ and classifying Du Val double planes we get $(d), (e)$. Finally, in §5 we collect some consequences of Theorem 0.1.

Acknowledgement. I would like to thank prof. Ciro Ciliberto, who suggested the problem, for his advise and encouragement. I also would like to thank Ingrid Bauer and Fabrizio Catanese for fruitful discussions and for their friendly hospitality when I was at the University of Bayreuth. I am indebted to Fabrizio Catanese who suggested to remove the hypothesis of regularity in a preliminary version of the main theorem.

1. NOTATION AND SET UP

Throughout the paper we will mean by surface (resp. curve) a projective algebraic surface (resp. curve) over the complex numbers and by a curve on a surface we will mean an effective non zero divisor on the surface. The symbol \equiv will denote the linear equivalence of divisors.

A smooth surface Y is ruled if there exists a surjective morphism f onto a curve whose general fibre is isomorphic to \mathbb{P}^1 . If each fibre of f is smooth one says that Y is geometrically ruled. Let Y' be a singular surface and $Y \rightarrow Y'$ a resolution of the singularities. Then we will say that Y' is ruled if Y is ruled.

Let C be a reduced curve singular at a point $p \in C$. The singularity is *non essential* if it is:

- either a double point,
- or a triple point which resolves to at most a double point after one blow up.

otherwise it is *essential*. Let p' be a point infinitely near to p . Then C has an $[r, r]$ -point at (p, p') if it has a point of multiplicity r at p which resolves to a point of multiplicity r at p' after one blowing up at p . We shall denote such singularity by $[p' \rightarrow p]$. Notice that an $[r, r]$ -point is an essential singularity if and only if $r \geq 3$.

We will use freely the theory of double covers referring to [1] for the details.

1.1. Surfaces with a 2-to-1 rational map. Let S be a smooth minimal surface of general type such that there is a generically finite rational map $\phi : S \dashrightarrow \Sigma$ of degree 2 onto a surface (for short, a 2-to-1 rational map).

Hence ϕ induces an involution σ on S which is a morphism since S is minimal of general type. The fixed locus $Fix(\sigma)$ is the union of a smooth reduced curve R_σ and k distinct points q_1, \dots, q_k . The canonical projection onto the quotient $\rho : S \rightarrow \Sigma_\sigma := S/\sigma$ is a double cover, i.e. a finite morphism of degree 2, branched along the smooth curve $B_\sigma = \rho(R_\sigma)$ and at the points $Q_i = \rho(q_i)$, $i = 1, \dots, k$. The only singularities of Σ_σ are the ordinary double points Q_1, \dots, Q_k .

Let $\hat{\pi} : \hat{S} \rightarrow S$ be the blow-up at q_1, \dots, q_k and let E_1, \dots, E_k be the exceptional (-1) -curves of $\hat{\pi}$. We denote by $\hat{\sigma}$ the induced involution on \hat{S} and the quotient $\hat{S}/\hat{\sigma}$ by $\hat{\Sigma}$. Furthermore, we denote $\hat{\pi}^{-1}(R_\sigma)$ by \hat{R} . Hence $Fix(\hat{\sigma}) = \hat{R} + E_1 + \dots + E_k$ and we get the following commutative diagram

$$\begin{array}{ccc} \hat{S} & \xrightarrow{\hat{\pi}} & S \\ \hat{\rho} \downarrow & & \downarrow \rho \\ \hat{\Sigma} & \xrightarrow{\eta} & \Sigma_\sigma \end{array}$$

where the morphism η is the minimal resolution of the singularities of Σ_σ and $\hat{\rho}$ is a double cover branched along the smooth curve $\hat{B} = \hat{B}' + C_1 + \dots + C_k$ where $\hat{B}' = \hat{\rho}(\hat{R})$ and $C_i = \hat{\rho}(E_i)$, $i = 1, \dots, k$. In particular, $C_i = \eta^{-1}(Q_i)$ is a (-2) -curve and $\hat{\Sigma}$ is smooth.

By the theory of double covers there exists $\hat{\Delta} \in Pic(\hat{\Sigma})$ such that $\hat{B} \in |2\hat{\Delta}|$ and $\rho_*\mathcal{O}_{\hat{S}} = \mathcal{O}_{\hat{\Sigma}} \oplus \mathcal{O}_{\hat{\Sigma}}(-\hat{\Delta})$. Therefore, $K_{\hat{S}} = \rho^*(K_{\hat{\Sigma}} + \hat{\Delta})$ and we have

$$\begin{aligned} H^i(\hat{S}, \mathcal{O}_{\hat{S}}(mK_{\hat{S}})) &\cong \\ &\cong H^i(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(m(K_{\hat{\Sigma}} + \hat{\Delta}))) \oplus H^i(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(mK_{\hat{\Sigma}} + (m-1)\hat{\Delta})) \end{aligned}$$

for each $i \geq 0$ and $m \geq 0$.

Now we assume that the bicanonical map of S factors through ϕ , then we have the following commutative diagram

$$\begin{array}{ccccc} \hat{S} & \xrightarrow{\hat{\pi}} & S & \xrightarrow{\varphi_{2K}} & S_2 \\ \hat{\rho} \downarrow & & \downarrow \rho & \searrow \phi & \uparrow \phi_2 \\ \hat{\Sigma} & \xrightarrow{\eta} & \Sigma_\sigma & \xrightarrow{\eta'} & \Sigma \end{array}$$

where $\eta' := \rho^{-1} \circ \phi$ is a birational map and φ_{2K} factors through ρ and $\hat{\rho}$.

Remark 1.1. In general φ_{2K} factors through ρ if and only if either $H^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + 2\hat{\Delta})) = 0$ or $H^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta})) = 0$.

Therefore, we know that in our situation one of the above vector spaces has to be trivial. In fact, in the following refined version of a proposition by M.Mendes

Lopes and R.Pardini (cfr.[12], Proposition 2.1) we will see that in our situation $H^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta})) = 0$.

Proposition 1.2. *Let S be a smooth minimal surface of general type and σ an involution acting on S . Let \hat{S} be the blow up of S at the isolated fixed point of σ and $\hat{\rho} : \hat{S} \rightarrow \hat{\Sigma} := \hat{S}/\hat{\sigma}$ the canonical projection onto the quotient. Denote by $\hat{\Delta} \in \text{Pic}(\hat{\Sigma})$ a divisor such that $K_{\hat{S}} = \hat{\rho}^*(K_{\hat{\Sigma}} + \hat{\Delta})$ and by k the number of isolated fixed points of σ . Then*

- a) $h^i(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta})) = 0$, for $i > 0$;
- b) let R_{σ} be the divisorial part of $\text{Fix}(\sigma)$, then
 - i) $k = K_S^2 - 2\chi(\mathcal{O}_S) + 6\chi(\mathcal{O}_{\hat{\Sigma}}) - 2h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta}))$
 - ii) $k = K_S.R_{\sigma} - 4\chi(\mathcal{O}_S) + 8\chi(\mathcal{O}_{\hat{\Sigma}})$
- c) Assume that $p_g(\hat{\Sigma}) = 0$, then the following three conditions are equivalent
 - i) the bicanonical map of S factors through $\hat{\rho}$;
 - ii) $h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta})) = 0$;
 - iii) $k = K_S^2 - 2\chi(\mathcal{O}_S) + 6\chi(\mathcal{O}_{\hat{\Sigma}})$.

Proof. a) We use the notation introduced before. As Σ_{σ} has at most canonical singularities, we have that $2K_S = \rho^*(2K_{\Sigma_{\sigma}} + R_{\sigma})$. Therefore, $2K_{\Sigma_{\sigma}} + R_{\sigma}$ is nef and big because $2K_S$ is nef and big, and so $2K_{\hat{\Sigma}} + \hat{B} = \eta^*(2K_{\Sigma_{\sigma}} + R_{\sigma})$ is nef and big.

On the other hand we have the following equality of \mathbb{Q} -divisors

$$K_{\hat{\Sigma}} + \hat{\Delta} = \frac{1}{2}(2K_{\hat{\Sigma}} + \hat{B}) + \frac{1}{2} \sum C_j$$

where $\frac{1}{2} \sum C_j$ is an effective \mathbb{Q} -divisor with zero integral part. Hence by the Kawamata-Viehweg vanishing theorem it follows that $h^i(\hat{\Sigma}, 2K_{\hat{\Sigma}} + \hat{\Delta}) = 0$, $i > 0$.

b) By a) and the Riemann-Roch formula we get:

$$h^0(\Sigma, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta})) = \chi(2K_{\hat{\Sigma}} + \hat{\Delta}) = \chi(\mathcal{O}_{\hat{\Sigma}}) + K_{\hat{\Sigma}}^2 + \frac{3}{2}K_{\hat{\Sigma}}.\hat{\Delta} + \frac{1}{2}\hat{\Delta}^2$$

and

$$\chi(\mathcal{O}_{\hat{S}}) = \chi(\mathcal{O}_{\hat{\Sigma}}) + \chi(K_{\hat{\Sigma}} + \hat{\Delta}) = 2\chi(\mathcal{O}_{\hat{\Sigma}}) + \frac{1}{2}(\hat{\Delta}^2 + \hat{\Delta}.K_{\hat{\Sigma}})$$

since $\rho_*\mathcal{O}_{\hat{S}} = \mathcal{O}_{\hat{\Sigma}} \oplus \mathcal{O}_{\hat{\Sigma}}(-\hat{\Delta})$. On the other hand we have

$$\begin{cases} k = K_S^2 - K_{\hat{S}}^2 = K_S^2 - 2(K_{\hat{\Sigma}} + \hat{\Delta})^2 \\ R_{\sigma}.K_S - k = (\hat{R} + \sum_{i=1}^k E_i).K_{\hat{S}} = 2\hat{\Delta}.(K_{\hat{\Sigma}} + \hat{\Delta}) \end{cases}$$

and so using the above equalities we get

$$\begin{cases} k = K_S^2 - 2\chi(\mathcal{O}_S) + 6\chi(\mathcal{O}_{\hat{\Sigma}}) - 2h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta})) \\ k = R_{\sigma}.K_S - 4\chi(\mathcal{O}_S) + 8\chi(\mathcal{O}_{\hat{\Sigma}}) \end{cases}$$

c) First of all recall that we have

$$p_g(S) = p_g(\hat{S}) = h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(K_{\hat{\Sigma}} + \hat{\Delta})) + h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(K_{\hat{\Sigma}})) = h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(K_{\hat{\Sigma}} + \hat{\Sigma})).$$

Therefore, if $p_g(S) > 0$ there is a non zero effective divisor $2D \in |2K_{\hat{\Sigma}} + \hat{B}|$ where $D \in |K_{\hat{\Sigma}} + \hat{\Delta}|$. Whence, if $p_g(S) > 0$ the bicanonical map of S factors through $\hat{\rho}$ and only if $h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta})) = 0$.

If $p_g(S) = 0$ then from $b)$ it follows

$$k = R_\sigma \cdot K_S + 4 = K_S^2 + 4 - 2h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta}))$$

where $k \geq 4$ since K_S is *nef*. Now assume that φ_{2K} factors through $\hat{\rho}$ and that $h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta})) \neq 0$. Then $h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{B})) = 0$ and we have

$$h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta})) = h^0(\hat{S}, \mathcal{O}_{\hat{S}}(2K_{\hat{S}})) = h^0(S, \mathcal{O}_S(2K_S)) = K_S^2 + 1$$

which by the above equality implies

$$k = K_S^2 + 4 - 2(K_S^2 + 1) = -K_S^2 + 2 \leq 1.$$

A contradiction. Whence, the bicanonical map of S factors through $\hat{\rho}$ if and only if $h^0(\hat{Y}, \mathcal{O}_{\hat{Y}}(2K_{\hat{Y}} + \hat{\Delta})) = 0$ and, by $b, i)$, the equivalence with $c, iii)$ is clear. \square

1.2. Canonical resolution. (cfr. [9], [1]) Let W_0 be a smooth surface. Assume that there exists a double cover $S_0 \rightarrow W_0$ branched along a reduced curve $B_0 \subset W_0$. Then S_0 is normal and it is smooth if and only if B_0 is smooth. If S_0 is singular the singularities of S_0 can be resolved in a natural way by the *canonical resolution*. Briefly, there is a commutative diagram

$$\begin{array}{ccccc} S^* & \xrightarrow{\quad\quad\quad} & S_0 & & \\ \rho' \downarrow & & \downarrow & & \\ W_s & \xrightarrow{\omega_s} & W_{s-1} & \xrightarrow{\omega_{s-1}} \cdots \xrightarrow{\omega_1} & W_0 \end{array}$$

such that, for each $i = 0, \dots, s-1$, ω_{i+1} is the blow up of $y_i \in W_i$, where $y_i \in B_i$ is a singular point of B_i .

Let m_i be the multiplicity of B_i at y_i and $\mathcal{E}_{i+1} = \omega_{i+1}^{-1}(y_i)$ the exceptional curve of ω_{i+1} , hence $B_{i+1} = (\omega_{i+1})^*(B_i) - 2[\frac{m_i}{2}]\mathcal{E}_{i+1}$ where $[\frac{m_i}{2}]$ is the greatest integer lesser than or equal to $\frac{m_i}{2}$. Furthermore, the curve $B_s \subset W_s$ is smooth, ρ' is a double cover branched along B_s and $S^* \rightarrow S_0$ is a birational morphism.

Let us denote by $\omega = \omega_1 \circ \cdots \circ \omega_s$ the composition and by $\mathcal{E}_{i+1}^* = \omega^*(y_i)$ the exceptional (-1) -cycle with reduced support $\omega^{-1}(y_i)$. Hence the following equalities hold

$$\begin{aligned} K_{W_s} &= \omega^*(K_{W_0}) + \sum \mathcal{E}_i^*; & \mathcal{E}_j^* \cdot \mathcal{E}_h^* &= -\delta_{j,h}; \\ B_s &= \omega^*(B_0) - \sum 2 \left[\frac{m_i}{2} \right] \mathcal{E}_i^* \end{aligned}$$

where $\delta_{j,h}$ is the Kronecker symbol.

Notice that S^* is also the canonical resolution of the double cover $S_i \rightarrow W_i$ branched along B_i , for each $i = 1, \dots, s$.

Lemma 1.3. *Let S, \hat{S} and $\hat{\Sigma}$ be as in Proposition 1.2. Let $\psi : \hat{\Sigma} \rightarrow \Sigma'$ be a birational morphism onto a smooth surface and consider a factorization of ψ in blow ups*

$$\psi : \hat{\Sigma} = \hat{\Sigma}_0 \xrightarrow{\psi_1} \hat{\Sigma}_1 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_t} \hat{\Sigma}_t = \Sigma'$$

For $i = 1, \dots, t$, denote by $\hat{y}_i \in \hat{\Sigma}_i$ the center of the blow up ψ_i and by $\hat{\mathcal{E}}_i = \psi_i^{-1}(\hat{y}_i)$ the exceptional curve of ψ_i . Moreover, let \hat{B}_i be the image of \hat{B} in $\hat{\Sigma}_i$. Set $\hat{B}_0 = \hat{B}$. Then for each $i \geq 1$

- (1) if \hat{B}_i has a singularity at a point z then either $\hat{y}_i = z$ or there exists $j < i$ such that $(\psi_i \circ \cdots \circ \psi_{j+1})(\hat{y}_j) = z$;
- (2) $\hat{\mathcal{E}}_i$ belongs to \hat{B}_{i-1} if and only if the multiplicity of \hat{B}_i at \hat{y}_i is odd;
- (3) \hat{B}_i is singular at \hat{y}_i ;
- (4) \hat{S} is the canonical resolution of the double cover of Σ' branched along \hat{B}_t .

Proof. We keep the notation from section 1.1. Since \hat{B} is smooth, 1) is clear. 2) For $i = 1, \dots, t$, let $\hat{\Delta}_i \in \text{Pic}(\hat{\Sigma}_i)$ denote $(\psi_i \circ \cdots \circ \psi_1)_*(\hat{\Delta})$. If $\hat{\mathcal{E}}_i \not\subset \hat{B}_{i-1}$ the multiplicity of $\hat{B}_i = \psi_{i*}(\hat{B}_{i-1})$ at \hat{y}_i is $\hat{\mathcal{E}}_i \cdot \hat{B}_{i-1} = 2\hat{\mathcal{E}}_i \cdot \hat{\Delta}_{i-1}$, an even number. On the other hand if $\hat{\mathcal{E}}_i \subset \hat{B}_{i-1}$ we have $\hat{\mathcal{E}}_i \cdot (\hat{B}_{i-1} - \hat{\mathcal{E}}_i) = 2\hat{\mathcal{E}}_i \cdot \hat{\Delta}_{i-1} - 1$, and so the multiplicity of $\hat{B}_i = \psi_{i*}(\hat{B}_{i-1}) = \psi_{i*}(\hat{B}_{i-1} - \hat{\mathcal{E}}_i)$ at \hat{y}_i is odd.

3) Let $\mathcal{E} \subset \hat{\Sigma}$ be a (-1) -curve and $E \subset \hat{S}$ a reduced and irreducible curve such that $\hat{\rho}(E) = \mathcal{E}$. If $\mathcal{E} \subset \hat{B}$ then $E^2 = \frac{1}{2}\mathcal{E}^2 = -\frac{1}{2}$, a contradiction. If $\mathcal{E} \cap \hat{B} = \emptyset$ then $E^2 = -1$ and $E \cdot E_i = 0, i = 1, \dots, k$, hence $\hat{\pi}(E) \subset S$ is a (-1) -curve, a contradiction. Therefore, $\mathcal{E} \not\subset \hat{B}$ and $\mathcal{E} \cdot \hat{B} \geq 1$, that is $\mathcal{E} \cdot \hat{B} \geq 2$ as $\hat{B} \equiv 2\hat{\Delta}$. In particular, it follows that \hat{B}_1 is singular at y_1 .

Now assume $i > 1$. By 1), 2) and the inductive hypothesis, $\hat{\mathcal{E}}_i \not\subset \hat{B}_{i-1}$ implies that \hat{B}_i has multiplicity $\hat{\mathcal{E}}_i \cdot \hat{B}_{i-1} \geq 2$ at \hat{y}_i while for $\hat{\mathcal{E}}_i \subset \hat{B}_{i-1}$ we get $\hat{\mathcal{E}}_i \cap (\hat{B}_{i-1} - \hat{\mathcal{E}}_i) \neq \emptyset$.

Hence \hat{B}_i is singular at \hat{y}_i if $\hat{\mathcal{E}}_i \not\subset \hat{B}_{i-1}$ and $\hat{B}_i = \psi_{i*}(\hat{B}_{i-1} - \hat{\mathcal{E}}_i)$ has multiplicity ≥ 1 at \hat{y}_i if $\hat{\mathcal{E}}_i \subset \hat{B}_{i-1}$. In the second case if $\hat{\mathcal{E}}_i \cdot (\hat{B}_{i-1} - \hat{\mathcal{E}}_i) = 1$ we can assume $\{\hat{y}_{i-1}\} = \hat{\mathcal{E}}_i \cap (\hat{B}_{i-1} - \hat{\mathcal{E}}_i)$ and still by induction we get that the strict transform $\bar{\mathcal{E}}_{i-1}$ (resp. $\bar{\mathcal{E}}_i$) of $\hat{\mathcal{E}}_{i-1}$ (resp. $\hat{\mathcal{E}}_i$) on $\hat{\Sigma}$ is a (-1) -curve ((-2) -curve) belonging (do not belonging) to \hat{B} such that $\bar{\mathcal{E}}_i \cdot \bar{\mathcal{E}}_{i-1} = 1$ and $\hat{B} \cdot \bar{\mathcal{E}}_{i-1} = 2$. Therefore, taking the pull back to \hat{S} of $\bar{\mathcal{E}}_{i-1}$ and then pushing it down to S we get a smooth rational curve with selfintersection greater than or equal to -1 . A contradiction.

Finally, for 4) it is easily seen that, since \hat{B} is smooth, 1), 2), 3) characterize the canonical resolution of the double cover of Σ_t branched along \hat{B}_t . \square

2. PROOF OF THEOREM 0.1: PART I

In this section and in the next one we will prove the implications $(a) \Rightarrow (b)$, $(a) \Rightarrow (c)$ of Theorem 0.1. Hence, throughout these two sections we will assume that S is a smooth minimal surface of general type such that the bicanonical map factors through a 2-to-1 map $\phi : S \dashrightarrow \Sigma$ onto a rational or ruled surface. We also assume that S does not present the standard case, in particular $K_S^2 \leq 9$.

Therefore, from section 1.1 we get the commutative diagram

$$\begin{array}{ccccc}
 \hat{S} & \xrightarrow{\pi} & S & \xrightarrow{\varphi_{2K}} & S_2 \\
 \hat{\rho} \downarrow & & \rho \downarrow & \searrow \phi & \uparrow \phi_2 \\
 \hat{\Sigma} & \xrightarrow{\hat{\eta}} & \Sigma_\sigma & \xrightarrow{\eta} & \Sigma
 \end{array}$$

where $\hat{\Sigma}$ is a rational or ruled surface since η, η' are birational maps. In particular, as $\hat{\Sigma}$ is smooth it is either ruled or \mathbb{P}^2 .

Proposition 2.1. *If $\hat{\Sigma} \cong \mathbb{P}^2$ then $q(S) = 0$ and \hat{B} is a smooth curve of degree 8 or 10. We have respectively $p_g(S) = 3$, $K_S^2 = 2$ and $p_g(S) = 6$, $K_S^2 = 8$. Moreover, K_S is ample.*

Proof. First of all notice that the involution σ induced by ϕ on S does not have isolated fixed points, otherwise there would be some (-2) -curve contained in \mathbb{P}^2 (cfr. (1.1)).

Hence $\hat{S} = S$, $\hat{\Sigma} = \Sigma_\sigma$ and ρ is a (finite) double cover. Therefore, \hat{B} is smooth and denoting by $2d$ the degree of $\hat{B} \equiv 2\hat{\Delta}$ we get:

$$9 \geq K_S^2 = 2(K_{\mathbb{P}^2} - \hat{\Delta})^2 = 2(d-3)^2,$$

$$h^i(S, \mathcal{O}_S(K_S)) = h^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-3)) + h^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)),$$

hence $2d \leq 10$. On the other hand we have $2d \geq 8$, since S is of general type. So $3 \leq d \leq 4$ and $q(S) = 0$, $p_g(S) = \frac{1}{2}d(d-3) + 1$.

We notice that there cannot be a (-2) -curve on S , since it would map to a (-1) -curve or a (-2) -curve in \mathbb{P}^2 , whence K_S is ample. \square

From now on we will assume that $\hat{\Sigma}$ is ruled. Let Σ_e be a geometrically ruled surface. We denote by C_0 a section of Σ_e such that the self intersection $C_0^2 = -e \leq 0$ is the smallest possible and by $\Gamma \cong \mathbb{P}^1$ we denote a fibre of the ruling. Recall that C_0 and Γ generate $\text{Pic}(\Sigma_e)$.

Hence there is a birational morphism $\varphi : \hat{\Sigma} \rightarrow \Sigma_e$ and setting $B = \varphi_*(\hat{B})$ we can write

$$B \equiv \xi C_0 + \left(\frac{1}{2}\xi e + \zeta\right)\Gamma$$

Following Xiao [18], we can assume φ to be such that

- †) $\xi = B \cdot \Gamma$ is minimal;
- ‡) the greatest multiplicity of the singularities of B is minimal, and the number of singularities of B with the greatest multiplicity is minimal, among all the choices satisfying condition (†);

where an $[r, r]$ -point is considered as a unique singularity of multiplicity strictly between r and $r+1$.

Remark. Let \hat{H} be the pull back to \hat{S} of a general $\Gamma \in |\Gamma|$. Hence $\varphi \circ \hat{\rho}|_{\hat{F}} : \hat{F} \rightarrow \Gamma$ is a double cover branched in $\Gamma \cdot B$ points. Therefore, $|\hat{H}|$ is a pencil of curves of genus $\frac{1}{2}(\Gamma \cdot B - 2)$. In particular, we assume $\xi \geq 8$ since S does not present the standard case.

The main result of this section is the following:

Theorem 2.2. *Let S be a smooth minimal surface of general type does not presenting the standard case and σ an involution acting on S such that the quotient S/σ is a ruled surface. Let \hat{S} be the blow up of S at the isolated fixed point of σ and $\hat{\rho} : \hat{S} \rightarrow \hat{\Sigma} = \hat{S}/\hat{\sigma}$ the projection onto the quotient. Let $\varphi : \hat{\Sigma} \rightarrow \Sigma_e$ be a birational morphism having the properties †) and ‡).*

Assume that the bicanonical map of S factors through $\hat{\rho}$. Then Σ_e is rational and only the following possibilities can occur:

- i) $\xi = 8$, $\zeta = 6$;
- ii) $\xi = 8$, $\zeta = 8 + 2i$, where $1 \leq i \leq 5$. The essential singularities of B are: $i+1$ $[5, 5]$ -points, possibly some 4-tuple points or $[3, 3]$ -points.

Remark 2.3. The idea of this theorem goes back to Xiao Gang. In fact, in [18] Proposition 6 he proves a weakly result, namely:

a) he further assumes the bicanonical map to be 2-to-1 onto a ruled surface and that $h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta})) = 0$;
 b) he claims that under these hypotheses Σ_e is rational and only the following possibilities can occur: (i), (ii) as above and

- iii) $\xi = 12$, $\zeta = 14$, and B has three $[7, 7]$ -points, possibly some non essential singularities;
- iv) $\xi = 16$, $\zeta = 18$, and B has three $[9, 9]$ -points, an 8-tuple point, possibly some non essential singularities.

Remark 2.4. In fact, Theorem 2.2 suffices to prove implication (a) \Rightarrow (b) of Theorem 0.1. In particular, we have that Σ_e is the Hirzebruch surface $\mathbb{F}_e = \mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$.

Remark 2.5. We will prove the above theorem in several steps:

- 1) we remark that looking carefully at the Xiao's proof it is easy to see that the argument still works if one suppose that the bicanonical map factors through a rational map of degree two onto a rational or ruled surface;
- 2) moreover, in our situation we have that $h^0(\hat{\Sigma}, \mathcal{O}_{\hat{\Sigma}}(2K_{\hat{\Sigma}} + \hat{\Delta})) = 0$ by Proposition 1.2;
- 3) therefore, we are now reduced to prove the following proposition:

Proposition 2.6. *In the hypotheses of Theorem 2.2, cases (iii), (iv) above do not occur.*

Notation. From now on we will refer to \hat{S} as a surface of type \mathcal{S}_I (resp. \mathcal{S}_{II} , \mathcal{S}_{III} , \mathcal{S}_{IV}) meaning that we consider \hat{S} associated to the commutative diagram

$$\begin{array}{ccc} & \hat{S} & \\ \hat{\rho} \downarrow & \searrow & \\ \hat{B} \subset \hat{\Sigma} & \xrightarrow{\varphi} & \mathbb{F}_e \supset B \end{array}$$

such that the morphism $\varphi : \hat{\Sigma} \rightarrow \mathbb{F}_e$ has the properties (†), (‡) and $B = \varphi_*(\hat{B})$ is as in Proposition 2.2, (i) (resp. 2.2, (ii), Remark 2.3, (iii), (iv)).

Let $p \in \mathbb{F}_e$ be a point. We denote by elm_p the elementary transformation centered at p , that is the result of blowing up p and then contracting the fibre of the ruling passing through p .

Lemma 2.7. *Let \hat{S} be a surface of type \mathcal{S}_{II} (resp. \mathcal{S}_{III} , \mathcal{S}_{IV}). Let $[p' \rightarrow p]$ be an $[r, r]$ -points of B such that $r = 5$ (resp. 7, 9). Let $\Gamma_p \in |\Gamma|$ be the fibre such that $p \in \Gamma_p$.*

Then, p' is infinitely near to Γ_p , Γ_p belongs to B and two distinct singular $[r, r]$ -points lie on distinct fibres. Finally, C_0 does not belong to B .

Proof. Suppose that p' is not infinitely near to Γ_p . Then $\Gamma_p \not\subset B$ and since $r = \frac{1}{2}\xi + 1 = \frac{1}{2}B \cdot \Gamma_p + 1$ we can apply $elm_{p'} \circ elm_p$ to obtain a new model with less singularities of maximal multiplicity. A contradiction.

Now the other claims are clear. \square

Lemma 2.8. *Let \hat{S} be a surface of type \mathcal{S}_{II} or \mathcal{S}_{III} or \mathcal{S}_{IV} . Then*

- if \hat{S} is of type \mathcal{S}_{II} then $0 \leq e \leq \frac{7+i}{4}$, $1 \leq i \leq 5$;

- if \hat{S} is of type \mathcal{S}_{III} or \mathcal{S}_{IV} then we can assume $e = 1$.

Proof. By Lemma 2.7 the curve B contains $i + 1$ (resp. 3) fibres if \hat{S} is of type \mathcal{S}_{II} (resp. \mathcal{S}_{III} or \mathcal{S}_{IV}) and $C_0 \not\subset B$.

Therefore, if \hat{S} is of type \mathcal{S}_{II} we get

$$i + 1 \leq C_0.B = -4e + 8 + 2i$$

that is

$$e \leq \frac{7+i}{4}$$

and analogously we get $e < 2$ if \hat{S} is of type \mathcal{S}_{III} or \mathcal{S}_{IV} .

Let us now suppose that \hat{S} is of type \mathcal{S}_{III} (resp. \mathcal{S}_{IV}) and $e = 0$. Then we can choose C_0 such that there exists a $[7,7]$ -point (resp. $[9,9]$ -point), say $[p' \rightarrow p]$, such that $p \in C_0$. Now performing elm_p we get a model with $e = 1$ and the same singularities. \square

For the remainder of the section we will assume that \hat{S} is of type \mathcal{S}_{III} or \mathcal{S}_{IV} . Therefore, by Lemma 1.3 \hat{S} is the canonical resolution of the double cover of \mathbb{F}_1 branched along a reduced curve $B = B' + \Gamma_1 + \Gamma_2 + \Gamma_3$ such that $\Gamma_i \in |\Gamma|$ and $B' \in |12C_0 + 17\Gamma|$ (resp. $B' \in |16C_0 + 21\Gamma|$).

We will denote by $[p'_i \rightarrow p_i]$ the $[7,7]$ -point (resp. $[9,9]$ -point) of B such that $p_i \in L_i$, $i = 1, 2, 3$.

Lemma 2.9. (a) For any curve $C \subset \mathbb{F}_1$ sitting in the linear system $|C_0 + \Gamma|$ we have that $\{p_1, p_2, p_3\} \not\subset C$;

(b) $p_i \notin C_0$, $i = 1, 2, 3$.

Proof. We give the proof for the case $B' \in |12C_0 + 17\Gamma|$, the other one is completely analogous. Suppose that there exists $C \in |C_0 + \Gamma|$ such that $\{p_1, p_2, p_3\} \subset C$. Then $21 \leq C.B = (C_0 + \Gamma).(12C_0 + 20\Gamma) = 20$ implies that C belongs to B . Hence C is tangent to Γ_i at p_i , for $i = 1, 2, 3$, a contradiction. Analogously we get $p_i \notin C_0$. \square

Since the $[p'_i \rightarrow p_i]$'s are the singularities of B with maximal multiplicity we have a factorization $\varphi = \pi_1 \circ \pi_2 \circ \cdots \circ \pi_6 \circ \varphi'$ such that π_i (resp. π_{3+i}) is the blow up at p_i (resp. p'_i), $i = 1, 2, 3$. We set $\hat{W} = \varphi'(\hat{S})$.

Let $\pi_0 : \mathbb{F}_1 \rightarrow \mathbb{P}^2$ be the morphism contracting C_0 to a point $p_0 \in \mathbb{P}^2$. Then $L_i = \pi_0(\Gamma_i)$ is a line passing through p_0 and by the above lemma p_1, p_2, p_3 , are non collinear points such that $p_i \neq p_0$, $i = 1, 2, 3$. (by abuse of notation we denote by the same letter the image of p_i in \mathbb{P}^2). Hence we have the commutative diagram

$$\begin{array}{ccc}
 \hat{S} & & \\
 \varphi' \downarrow & \searrow \varphi & \\
 \hat{W} & \xrightarrow{\pi_1 \circ \cdots \circ \pi_6} & \mathbb{F}_1 \\
 & \searrow \pi & \downarrow \pi_0 \\
 & & \mathbb{P}^2
 \end{array}$$

where $\pi = \pi_0 \circ \pi_1 \circ \cdots \circ \pi_6 : \hat{W} \rightarrow \mathbb{P}^2$ is the composition. We set $\mathcal{E}_0^* = \pi^*(p_0)$, $\mathcal{E}_i^* = \pi^*(p_i)$, $\mathcal{E}_{3+i}^* = \pi^*(p'_i)$, $i = 1, 2, 3$.

Lemma 2.10. *Let L be a line in \mathbb{P}^2 . Then*

$$h^0(\hat{W}, \mathcal{O}_{\hat{W}}(\pi^*(5L) - \mathcal{E}_0^* - \sum_{i=1}^6 2\mathcal{E}_i^*)) = 2$$

and the general element

$$D \in |\pi^*(5L) - \mathcal{E}_0^* - \sum_{i=1}^6 2\mathcal{E}_i^*|$$

is a smooth and irreducible rational curve on \hat{W} such that $D^2 = 0$.

In particular, $|D|$ defines surjective morphism $f : \hat{W} \rightarrow \mathbb{P}^1$ such that the general fibre is isomorphic to \mathbb{P}^1 .

Proof. Let \mathcal{C}_i be the conic in \mathbb{P}^2 passing through p_1, p_2, p_3 , tangent to L_j, L_k , $\{i, j, k\} = \{1, 2, 3\}$. Then \mathcal{C}_i is smooth since p_1, p_2, p_3 are not collinear and $p_i \neq p_0$. Let $\overline{D}_i \subset \hat{W}$ be the strict transform of the curve $D_i = 2\mathcal{C}_i + L_i$, $i = 1, 2, 3$. Hence $\overline{D}_i, \overline{D}_j$ do not have common components if $i \neq j$ and $\overline{D}_i \in |D|$. In particular, $h^0(\hat{W}, \mathcal{O}_{\hat{W}}(D)) \geq 2$ and $|D|$ does not have fixed part.

A straightforward calculation yields $D^2 = 0, D.K_{\hat{W}} = -2$. Therefore, the rational map f defined by $|D|$ is a surjective morphism onto a curve and $D \in |a\mathcal{L}|$, where \mathcal{L} is a general fibre of f .

On the other hand $\mathcal{E}_0^*.D = \mathcal{E}_0^*.(\pi^*(5L) - \mathcal{E}_0^* - \sum_{i=1}^6 2\mathcal{E}_i^*) = 1$. Therefore, $a = 1$ and a standard argument completes the proof. \square

Now we are ready to prove Proposition 2.6 and then Theorem 2.2 by Remark 2.5.

Proof of Proposition 2.6. Let D be as in the above lemma. Denote by $\pi' = \pi_1 \circ \dots \circ \pi_6 : \hat{W} \rightarrow \mathbb{F}_1$ the composition such that $\varphi = \pi' \circ \varphi'$. Hence, a straightforward calculation yields $D \in |\pi'^*(4C_0 + 5\Gamma) - \sum_{i=1}^6 2\mathcal{E}_i^*|$.

Assume that \hat{S} is of type \mathcal{S}_{III} . Then $B \equiv 12C_0 + 20\Gamma$ and \hat{S} is the canonical resolution of the double cover of \mathbb{F}_1 branched along B . Hence we have

$$\varphi'_*(\hat{B}) = \pi'^*(12C_0 + 20\Gamma) - \sum_{i=1}^3 6\mathcal{E}_i^* - \sum_{i=4}^6 8\mathcal{E}_i^*$$

since the $[p'_i \rightarrow p_i]$'s are $[7, 7]$ -points. Therefore, we get

$$D.(\varphi'_*(\hat{B})) = (4C_0 + 5\Gamma).(12C_0 + 20\Gamma) - 12 \cdot 3 - 16 \cdot 3 = 8$$

which is a contradiction. Indeed in this case there is a birational morphism $\tilde{\varphi} : \hat{W} \rightarrow \mathbb{F}_e$ such that $\tilde{\varphi}_*(D)$ is a ruling of \mathbb{F}_e and so if we consider the morphism $\tilde{\varphi} \circ \varphi' : \hat{S} \rightarrow \mathbb{F}_e$ we get $((\tilde{\varphi} \circ \varphi')_*(\hat{B})).(\tilde{\varphi}_*(D)) = 8 < 12$. But we are assuming that φ has the property (\dagger) .

An analogous argument shows that \hat{S} can not be of type \mathcal{S}_{IV} . \square

3. PROOF OF THEOREM 0.1: PART II

In this section we will prove that if \hat{S} is of type \mathcal{S}_I (resp. \mathcal{S}_{II}) then S is the minimal model of a Du Val double plane. Therefore, we get the implication (a) \Rightarrow (c) of Theorem 0.1.

Proposition 3.1. *Let S be such that \hat{S} is a surface of type \mathcal{S}_I . Then there exists a birational morphism $\psi : \hat{\Sigma} \rightarrow X$ onto a smooth surface such that setting $G := \psi_*(\hat{B})$ we have:*

- a) *either $X \cong \mathbb{P}^2$ and G is a reduced curve of degree 10 with possibly some $[3, 3]$ -points and no other essential singularities;*
- b) *or $X \cong \mathbb{F}_2$ and $G = C_0 + G'$, where $G' \cap C_0 = \emptyset$ and G' is a reduced curve in the linear system $|7C_0 + 14\Gamma|$ with at most non essential singularities;*
- c) *or $X \cong \mathbb{P}^2$ and $G = G' + L_1$ where L_1 is a line and G' is a reduced curve of degree 11. In this case G has the following essential singularities: a 4-tuple point and a $[5, 5]$ -point on L_1 , possibly some $[3, 3]$ -points.*

In particular, \hat{S} and S are respectively the canonical resolution and the smooth minimal model of a Du Val double plane.

Proof. This was already partially proved by Xiao Gang. In fact, we have a morphism $\varphi : \hat{\Sigma} \rightarrow \mathbb{F}_e$ such that $B \equiv 8C_0 + (4e + 6)\Gamma$ and by [18] Proposition 7, either $e = 1$ or $e = 2$, hence either $B \equiv 8C_0 + 10\Gamma$ or $B \equiv 8C_0 + 14\Gamma$. Still by [18] Proposition 7 the essential singularities of B are possibly 4-tuple points or $[3, 3]$ -points.

If $e = 1$ let cont_{C_0} be the morphism which contracts the (-1) -section to a point $p \in \mathbb{P}^2$. We obtain a morphism onto \mathbb{P}^2

$$\psi := \text{cont}_{C_0} \circ \varphi : \hat{\Sigma} \rightarrow \mathbb{P}^2$$

such that $G := \psi_*(\hat{B})$ is a curve of degree 10. Notice that $C_0.B = 2$ and so G has either a triple point or a double point at p depending on $C_0 \subset B$ or not. Thus the essential singularities of G are possibly 4-tuple points or $[3, 3]$ -points. Suppose that G has a 4-tuple point at say q . Then the pull-back to \hat{S} of the pencil $|L - q|$ of lines through q is a pencil of curves of genus 2, a contradiction.

If $e = 2$ we have $C_0.B = -2$, thus $C_0 \subset B$ and $B = C_0 + B'$ where $B' := B - C_0$ is a curve such that $B' \cap C_0 = \emptyset$ since B is reduced.

Suppose that B has two 4-tuple points, say p, q . Then the pull-back to \hat{S} of the pencil $|C - p - q|$, where $C \equiv C_0 + 2\Gamma$, is a pencil of curves of genus two, a contradiction.

If B has only non essential singularities we set $X = \mathbb{F}_2$ and $G = B$ (case (b)). If B has a 4-tuple point at say p we consider the projection from $p \in \mathbb{F}_2$ onto the plane, i.e. perform an elementary transformation centered at p and then contract the proper transform of C_0 . Since we blow up a singular point of B , by Lemma 1.3 we get a birational morphism $\psi : \Sigma \rightarrow X = \mathbb{P}^2$. We set $G = \psi_*(\hat{B})$, hence G is a curve of degree 10 which possibly has some $[3, 3]$ -points and no other essential singularities (case (a)). If B has only $[3, 3]$ -points as essential singularities, let $[p' \rightarrow p]$ be one of them. Hence, projecting from $p \in \mathbb{F}_2$ onto the plane we get a birational morphism $\psi : \hat{\Sigma} \rightarrow \mathbb{P}^2$ such that $G := \psi(\hat{B})$ is a curve of degree 12. Moreover, it is easily seen that $G = G' + L_1$, where L_1 is a line, and the essential singularities of G are: a 4-tuple point and a $[5, 5]$ -point lying on L_1 , possibly some $[3, 3]$ -points. Notice that L_1 is the image of the exceptional curve arising from p .

Finally, by Lemma 1.3 \hat{S} is the canonical resolution of the double cover of X branched along G . \square

Proposition 3.2. *Let S be such that \hat{S} is a surface of type \mathcal{S}_{II} . Then there exists $n \geq 2$ and a birational morphism $\psi : \hat{\Sigma} \rightarrow \mathbb{P}^2$ such that setting $G = \psi_*(\hat{B})$ we have:*

- a) $G = G' + \sum_{i=1}^n L_i$, where L_1, \dots, L_n are distinct lines passing through a point γ in \mathbb{P}^2 and $G' \in |(10+n)L|$ is a reduced curve;
- b) the essential singularities of G are a $(2n+2)$ -tuple point at γ , a $[5, 5]$ -point $[p'_i \rightarrow p_i]$ such that $p_i \in L_i$, $i = 1, \dots, n$, possibly some 4-tuple points or $[3, 3]$ -points.

Therefore,

- c) \hat{S} respectively S are the canonical resolution and the smooth minimal model of a Du Val double plane of type \mathcal{D}_n with $n \geq 2$.

Proof. We have a morphism $\varphi : \hat{\Sigma} \rightarrow \mathbb{F}_e$ such that $\varphi_*(\hat{B}) = B = B_1 + \Gamma_1 + \dots + \Gamma_n$, $n \in \{2, \dots, 6\}$, where $\Gamma_1, \dots, \Gamma_n$ are pairwise distinct fibres and $B \equiv 8C_0 + (4e + 8 + 2(n-1))\Gamma$.

The essential singularities of B are a $[5, 5]$ -point $[p'_i \rightarrow p_i]$ such that $p_i \in \Gamma_i$, $i = 1, \dots, n$, possibly some 4-tuple points or $[3, 3]$ -points.

By Lemma 2.7 and 2.8, we know that $C_0 \not\subset B$ and $0 \leq e \leq \frac{6+n}{4}$. In particular, $e \leq 3$ since $n \leq 6$.

If $e = 1$ let $\text{cont}_{C_0} : \mathbb{F}_1 \rightarrow \mathbb{P}^2$ be the birational morphism which contracts C_0 to a point γ in \mathbb{P}^2 . We denote by the same letter the image of p_i in \mathbb{P}^2 . Then $L_i := \text{cont}_{C_0}(\Gamma_i)$ is a line passing through γ and $G' = \text{cont}_{C_0*}(B_1)$ is a reduced curve having an m -tuple point at γ where $m = B_1.C_0 = n+2$, a $[4, 4]$ -point $[p'_i \rightarrow p_i]$ such that $p_i \in L_i$ and p'_i is infinitely near to L_i , $i = 1, \dots, n$, possibly some 4-tuple points or $[3, 3]$ -points.

We set $G = G' + L_1 + \dots + L_n$ and $\psi = \text{cont}_{C_0} \circ \varphi$. Hence a straightforward calculation shows that $G \in |(10+2n)L|$ and $\psi_*(\hat{B}) = G$. Whence (a), (b) follow and by Lemma 1.3 we get (c).

If $e = 2$ we can assume that $p_1 \notin C_0$. In fact, if were $p_i \in C_0$, for $i = 1, \dots, n$, then it would be $5n \leq C_0.B = -16 + 16 + 2(n-1) = 2n-2$, a contradiction.

Let us perform the elementary transformation $\text{elm}_{p_1} : \mathbb{F}_2 \dashrightarrow \mathbb{F}_1$ and consider the curve

$$B' = B'_1 + \Gamma'_1 + \dots + \Gamma'_{n-1} + \Gamma'_n$$

where Γ'_i (resp. B'_1) is the proper transform of Γ_i , $i = 2, \dots, n$, (resp. B_1) and Γ'_1 is the (image of the) exceptional curve arising from p_1 .

Then the proper transform C'_0 of C_0 is the (-1) -section and $C'_0 \not\subset B'$. As we blow up at a singular point of B with odd multiplicity, by Lemma 1.3 there exists a birational morphism $\varphi' : \hat{\Sigma} \rightarrow \mathbb{F}_1$ such that $\varphi'(\hat{B}) = B'$.

Moreover, a straightforward calculation shows that $B'.C'_0 = 2n+2$ and B' has the same singularities as B . Therefore we conclude as above.

If $e = 3$ we have $n = 6$ since $n \geq 4e - 6 = 6$ and so $C_0.B = 6$. Hence we see that $p_i \notin C_0$, $i = 1, \dots, 6$. Consider the birational map $\text{elm}_{p_1} \circ \text{elm}_{p_2} : \mathbb{F}_3 \dashrightarrow \mathbb{F}_1$.

Then as above we have a morphism $\varphi' : \hat{\Sigma} \rightarrow \mathbb{F}_1$ such that $\varphi'(\hat{B}) = B'$, where $B' = B'_1 + \Gamma'_1 + \dots + \Gamma'_6$ is composed by the proper transforms of $B_1, \Gamma'_3, \dots, \Gamma'_6$ and by the exceptional curves Γ'_1, Γ'_2 arising from p_1, p_2 .

Also in this case we get $C'_0.B' = 2n+2$ and B' has the same singularities as B . Therefore we conclude as above.

If $e = 0$ we argue as in the other cases. □

4. DU VAL DOUBLE PLANES

We are going to complete the proof of Theorem 0.1. In particular, we will prove implication (c) \Rightarrow (b) and assuming (c) we will show that S is regular unless $p_g(S) = q(S) = 1$.

Hence, throughout this section we will assume that S is a minimal surface of general type which is the smooth minimal model of a Du Val double plane X' and such that does not present the standard case.

We will denote by S^* the canonical resolution of such a double plane, so we have the following commutative diagram

$$\begin{array}{ccc}
 & & S \\
 & \nearrow \pi & \\
 S^* & \xrightarrow{\quad} & X' \\
 \downarrow \tilde{\rho} & & \downarrow \rho \\
 W_s & \xrightarrow{\omega} & W_0 = X
 \end{array}$$

(Note: In the original image, there is a dotted arrow from X' to S and a dotted arrow from X' to W_0 labeled ϕ .)

where X is either \mathbb{P}^2 or \mathbb{F}_2 according to the type of X' (cfr. introduction) and $\rho, \tilde{\rho}$ are double covers branched along $G, G_s = \omega^*(G) - \sum 2[\frac{m_i}{2}]\mathcal{E}_i^*$ respectively (cfr. (1, 2)). Furthermore, there is an involution σ^* on S^* induced by $\tilde{\rho}$ whose fixed locus is the divisor $R^* := \tilde{\rho}^{-1}(G_s)$. We denote by $\Delta \in \text{Pic}(X)$ (resp. $\Delta_s \in \text{Pic}(W_s)$) a divisor such that $G \in |2\Delta|$ (resp. $G_s \in |2\Delta_s|$).

Notation. Let X' be a Du Val double plane of type \mathcal{D}_n . We denote by δ_1 the number of $[3, 3]$ -points of the branch curve G , whereas by δ_2 we denote the number of 4-tuple points.

Furthermore, if $n > 0$ we denote by $[p'_i \rightarrow p_i]$ the $[5, 5]$ -point of G such that $p_i \in L_i, i = 1, \dots, n$, whereas if $\delta_1 > 0$ (resp. $\delta_2 > 0$) we denote by $[q'_j \rightarrow q_j]$ (resp. r_j) a $[3, 3]$ -point (resp. 4-tuple point) of $G, j = 1, \dots, \delta_1$ (resp. δ_2).

Lemma 4.1. *Let S^* be the canonical resolution of a Du Val double plane of type \mathcal{D}_n . Then*

- i) $p_g(S^*) - q(S^*) = 6 - n - \delta_1 - \delta_2$
- ii) $K_{S^*}^2 = 8 - 2n - 2\delta_1 - 2\delta_2$

Moreover,

- iii) $n + \delta_1 + \delta_2 \leq 6$;
- iv) if $n \leq 1$, then $\delta_2 \leq n$.

Proof. By [9] we have

$$\begin{aligned}
 \chi(S^*) &= \frac{1}{2}(K_{\mathbb{P}^2} + \Delta) \cdot \Delta + 2\chi(\mathbb{P}^2) - \frac{1}{2} \sum \left[\frac{m_i}{2} \right] \left(\left[\frac{m_i}{2} \right] - 1 \right) = \\
 &= \frac{1}{2}(2 + n) \cdot (5 + n) + 2 - \frac{1}{2}((n + 1)n + (2 + 6)n + 2\delta_1 + 2\delta_2) = \\
 &= 7 - n - \delta_1 - \delta_2
 \end{aligned}$$

and

$$\begin{aligned} K_{S^*}^2 &= 2(K_{\mathbb{P}^2} + \Delta)^2 - 2 \sum \left(\left\lfloor \frac{m_i}{2} \right\rfloor - 1 \right)^2 = \\ &= 2(2+n)^2 - 2(n^2 + (1+4)n + \delta_1 + \delta_2) = \\ &= 8 - 2n - 2\delta_1 - 2\delta_2. \end{aligned}$$

Since $\chi(S^*) = \chi(S) \geq 1$, i implies $n + \delta_1 + \delta_2 \geq 6$.

Finally, assume that $n = 0$ (resp. $n = 1$) and $\delta_2 \geq 1$ (resp. ≥ 2). Let $\mathcal{C} \subset W_s$ be the strict transform of a general line (resp. conic) passing through r_1 (resp. r_1, r_2, p_1, p'_1) and $\tilde{\mathcal{C}}$ its pull back to S^* . Then $|\tilde{\mathcal{C}}|$ is a pencil of curves of genus 2. A contradiction. \square

Notice that if $n = 1$ then γ is a 4-tuple point, hence γ may be infinitely near to p'_1 .

Recall that ω factors as $\omega_s \circ \dots \circ \omega_1$ where ω_{i+1} is the blow up of $y_i \in W_i$ with exceptional curve \mathcal{E}_{i+1} , $i = 0, \dots, s-1$.

Lemma 4.2. *Let S^* be the canonical resolution of a Du Val double plane of type \mathcal{D}_n and let C be a reduced and irreducible curve on W_s . Then*

- (1) *C is a (-2) -curve contained in G_s such that $\omega(C) = p$ is a point if and only if there exists $i \in \{0, \dots, s-2\}$ such that G_i has an $[r, r]$ -point at y_i with $r \geq 3$ odd, $(\omega_{i+2} \circ \dots \circ \omega_s)(C) = \mathcal{E}_{i+1}$ and $(\omega_1 \circ \dots \circ \omega_{i-1})(y_i) = p$ (or $y_0 = p$ if $i = 0$).*
- (2) *Assume that $n \geq 1$. Then C is a (-2) -curve contained in G_s such that $\omega(C) = L$ is a line passing through γ (resp. p_i , $i = 1, \dots, n$) if and only if $L \in \{L_1, \dots, L_n\}$.*

Proof. (1) is straightforward.

(2). If $\omega(C) = L$ is a line then C is the strict transform of L , because it is reduced and irreducible. Notice that since $L_i(G - L_i) = 2n + 9$ the only singular points of G lying on, or infinitely near to, L_i are γ, p_i, p'_i .

If L is a line passing through γ we can assume $y_0 = \gamma$ and hence we have that

$$C = \omega^*(L) - \mathcal{E}_1^* - \sum_{i \geq 2} c_i \mathcal{E}_i^*$$

where $c_i = 1$ if and only if y_{i-1} lies on, or is infinitely near to, L and $c_i = 0$ otherwise. Thus, $L^2 - C^2 = 1 - 1 - \sum c_i$ and so $C^2 = -2$ if and only if there are exactly two c_i 's which are non zero. Therefore, we get

$$-2 = C.G_s = L.G - (2n + 2) - 2\left[\frac{m_j}{2}\right] - 2\left[\frac{m_k}{2}\right]$$

where $j, h \in \{2, \dots, s\}$ are such that $\left[\frac{m_j}{2}\right] + \left[\frac{m_k}{2}\right] = 5$. It is easy to check that the only possibility is $L \in \{L_1, \dots, L_n\}$.

If $p_i \in L$ then L is tangent to L_i , since $L \subset G$, and so $L = L_i$. \square

Now for the proof of the main theorem we consider three cases:

- A) X' is of type \mathcal{D}_n with $n \geq 2$;
- B) X' is of type \mathcal{D}_n with $n < 2$;
- C) X' is of type \mathcal{D} or \mathcal{B} .

Case A). X' is of type \mathcal{D}_n , $n \geq 2$. As $2n+2 \geq 6$ we can assume $y_0 = \gamma$. Moreover, since the $[5, 5]$ -points are the singularities of G with the greatest multiplicity lower than $2n+2$ we can assume $y_i = p_i$ and $y_{n+i} = p'_i$, $i = 1, \dots, n$. Finally, we assume $y_{2n+i} = r_i$ for $i = 1, \dots, \delta_2$ and $y_{2n+\delta_2+j} = q_j$ (resp. $y_{2n+\delta_2+\delta_1+j} = q'_j$) for $j = 1, \dots, \delta_1$.

Proposition 4.3. *Let S be the smooth minimal model of a Du Val double plane of type \mathcal{D}_n with $n \geq 2$. Then*

a)

$$\begin{aligned} p_g(S) - q(S) &= 6 - n - \delta_1 - \delta_2 \\ K_S^2 &= 8 - \delta_1 - 2\delta_2 \end{aligned}$$

b) *there is a rational pencil $|H|$ on S such that:*

- i) *the general member $H \in |H|$ is a smooth hyperelliptic curve of genus 3;*
- ii) *$|H|$ has n double curves;*
- iii) *$|H|$ does not have base points.*

c) *The bicanonical map of S factors through ρ and it induces the hyperelliptic involution on the general $H \in |H|$.*

Proof. Let L be a general line in \mathbb{P}^2 passing through y_0 and $\tilde{L} = \omega^*(L) - \mathcal{E}_1^*$ its strict transform. Let $\tilde{H} = \tilde{\rho}^*(\tilde{L})$ be the pull back of \tilde{L} to S^* . Therefore, $\tilde{\rho}|_{\tilde{H}} : \tilde{H} \rightarrow \tilde{L}$ is a double cover branched in $\tilde{L}.G_s = 8$ points and \tilde{H} is a smooth hyperelliptic curve of genus 3. Moreover, $|\tilde{H}|$ is a rational pencil such that

$$\tilde{H}^2 = 0, \quad \tilde{H}.K_{S^*} = 4, \quad \tilde{H}.R^* = 8.$$

For each $i = 1, \dots, n$ we set $\tilde{L}_i = \omega^*(L_i) - \mathcal{E}_1^* \in |\tilde{L}|$ and we denote respectively by $C_i = \omega^*(L_i) - \mathcal{E}_1^* - \mathcal{E}_{i+1}^* - \mathcal{E}_{i+n+1}^*$ the strict transform of L_i and by $C_{n+i} = \mathcal{E}_{i+1}^* - \mathcal{E}_{i+n+1}^*$ the strict transform of $\mathcal{E}_{i+1} = \omega_{i+1}^{-1}(y_i)$ on W_s .

Hence we have $\tilde{L}_i = C_i + C_{n+i} + 2\mathcal{E}_{n+1+i}^*$ and C_1, \dots, C_{2n} are (-2) -curves belonging to G_s , by Lemma 4.2. Therefore, setting $\tilde{H}_i = \tilde{\rho}^*(\tilde{L}_i)$ and $E_i = \tilde{\rho}^{-1}(C_i)$ we get $\tilde{H}_i = 2E_i + 2E_{n+i} + 2\tilde{\rho}^*(\mathcal{E}_{n+1+i}^*)$, i.e. \tilde{H}_i is a double curve.

If $\delta_1 > 0$, by Lemma 4.2 there are δ_1 more (-2) -curves $C_{2n+1}, \dots, C_{2n+\delta_1}$ arising from the $[3, 3]$ -points which belong to G_s . We set $E_i = \tilde{\rho}^{-1}(C_i)$, $i = 2n+1, \dots, 2n+\delta_1$.

Notice that the E_i 's are (-1) -curves on S^* and since G_s is smooth they are pairwise disjoint. Moreover, it is easily seen that $C_i.\tilde{L} = E_i.\tilde{H} = 0$.

Since S is minimal of general type, the birational morphism $\pi : S^* \rightarrow S$ factors as $\pi_2 \circ \pi_1$ where $\pi_1 : S^* \rightarrow S'$ contracts (exactly) the E_i 's. Hence, by Lemma 4.1 we get

$$p_g(S') - q(S') = p_g(S^*) - q(S^*) = 6 - n - \delta_1 - \delta_2$$

$$K_{S'}^2 = K_{S^*}^2 + 2n + \delta_1 = 8 - \delta_1 - 2\delta_2$$

and so the following table for $(\chi(S') - 1, K_{S'}^2)$:

$\chi(S') - 1$	$K_{S'}^2$														
4															
3															
2															
1															
0	0	2	4	6	8	6	4	2	0	2	4	6	8	6	4
		1	3	5	7	5	3	1		3	5	7	5	3	1
			2	4	6	4	2			4	6	4	2		
				3	5	3	1			5	3	1			
					4	2				6	4	2			
						5	3			7	5	3			
							6	4		8	6	4			
								5	3						
									4	2					
										3	1				
											2				
												1			
													0		

where $K_{S'}^2 = 8$ if and only if $\delta_1 = \delta_2 = 0$ if and only if G has neither 4-tuple points nor $[3,3]$ -points and the arrow \swarrow (resp. \downarrow) means that one imposes one more 4-tuple point (resp. $[3,3]$ -point) to G .

Set $H' = \pi_{1*}(\tilde{H})$. Then the general member of $|H'|$ is a smooth hyperelliptic curve of genus 3 because $\tilde{H}.E_i = 0$, and $H'_j := \pi_{1*}(\tilde{H}_j)$ is a double curve for each $j = 1, \dots, n$. In particular, we have

$$H'^2 = 0, \quad K_{S'}.H' = 4, \quad R'.H' = 8.$$

Notice that σ^* induces an involution σ' on S' which is a morphism and whose fixed locus is union of the smooth curve $R' := \pi_{1*}(R^*)$ and the points $\pi_1(E_1), \dots, \pi_1(E_{2n+\delta_1})$.

Let H denote the image on S of a general $H' \in |H'|$. Suppose that $S' = S$. Hence (a) and (b) follow. Moreover, as S is minimal and W_s is rational we can apply Proposition 1.2. Therefore, we have $h^i(W_s, \mathcal{O}_{W_s}(2K_{W_s} + \Delta_s)) = 0$, $i > 0$, and for (c) it suffices to show that $h^0(W_s, \mathcal{O}_{W_s}(2K_{W_s} + \Delta_s)) = 0$.

On the other hand by the Riemann-Roch formula we get

$$\begin{aligned} h^0(W_s, \mathcal{O}_{W_s}(2K_{W_s} + \Delta_s)) &= \chi(2K_{W_s} + \Delta_s) = \\ &= \frac{1}{2}(2K_{\mathbb{P}^2} + \Delta).(K_{\mathbb{P}^2} + \Delta) - \frac{1}{8} \sum (m_i - 4)(m_i - 2) + 1 \\ &= \frac{1}{2}(n^2 + n - 2) - \frac{1}{8}(4n^2 + 4n) + 1 = 0 \end{aligned}$$

whence the bicanonical map of S factors through ρ .

So it remains to prove that $S' = S$. Suppose to the contrary that $S' \neq S$, then $\pi_2 : S' \rightarrow S$ is not the identity and there is a (-1) -curve $E \subset S'$ contracted by π_2 to a point.

First of all we claim that $E.H' = 0$. In fact, $E.H' \geq 0$ since $|H'|$ is a pencil.

If $E.H' > 0$ then $E.H' \geq 2$ since $|H'|$ has $n \geq 2$ double curves. Then we get $H^2 \geq 4$, $H.K_S \leq 2$ and so the Hodge Index theorem implies that $K_S^2 = 1$ and H is numerically equivalent to $2K_S$. Observe that in this case X' is of type \mathcal{D}_2 with $\delta_1 = 0, \delta_2 = 4$. In particular, the involution σ' acting on S' has 4 isolated fixed points.

As $K_S^2 = 1$ then π_2 contracts exactly E , and $H^2 = 4$ implies $E.H' = 2$. Now we have to consider two cases: either E belongs to R' or not.

If E belongs to R' then $q := \pi_2(E)$ is an isolated fixed point of the induced involution σ on S since R' is smooth. Moreover we get $K_S.R = \frac{1}{2}H.R = 3$ where we denote by $R := \pi_{2*}(R')$ the divisorial part of $\text{Fix}(\sigma)$. A contradiction, indeed σ has 5 isolated fixed points and so by Proposition 1.2 we get $K_S.R = 1$.

If E does not belong to R' then $2K_S.R = H.R \geq 8$. On the other hand, in this case σ has 4 isolated fixed points and so by Proposition 1.2 it follows $4 = K_S.R + 4$. A contradiction.

Therefore, $H'.E = 0$.

Let $\bar{E} \in S^*$ be the strict transform of E , we set $\mathcal{E} = \tilde{\rho}(\bar{E})$. Therefore, we have $\mathcal{E}.\tilde{L} = \bar{E}.\tilde{H} = E.H' = 0$ and then \mathcal{E} is a component of a curve $\tilde{L}_E \in |\tilde{L}|$. In particular, \mathcal{E} is a smooth rational curve and $\tilde{\rho}|_{\bar{E}} : \bar{E} \rightarrow \mathcal{E}$ is either a double cover or an isomorphism. We consider the two cases separately.

If $\tilde{\rho}|_{\bar{E}}$ is an isomorphism. We have $\tilde{\rho}^*(\mathcal{E}) = \bar{E} + \tilde{E}$ where $\bar{E} \cong \tilde{E}$ (possibly $\bar{E} = \tilde{E}$).

If $\bar{E} = \tilde{E}$ then $\mathcal{E} \subset G_s$ and thus $\bar{E} \cap E_i = \emptyset, i = 1, \dots, 2n + \delta_1$ since G_s is smooth. Hence $-1 = E^2 = \frac{1}{2}\mathcal{E}^2$. By Lemma 4.2 we get a contradiction.

Therefore, $\bar{E} \neq \tilde{E}$. In this case we have $G_s|_{\mathcal{E}} = 2z$ where $z \in \text{Pic}(\mathcal{E})$ and $\bar{E}^2 = \tilde{E}^2 = \mathcal{E}^2 - \deg(z)$. Then either $G_s \cap \mathcal{E} = \emptyset$ or G_s and \mathcal{E} are tangent at each intersection point. In particular $E \cap E_j = \mathcal{E} \cap C_j = \emptyset, j = 1, \dots, 2n + \delta_1$ since $\mathcal{E} \cap C_j \neq \emptyset$ implies that both \mathcal{E} and C_j belong to \tilde{L}_E . Hence \bar{E}, \tilde{E} are (-1) -curves and either $\mathcal{E}^2 = 0$ or $\deg(z) = 0$, since $\mathcal{E}^2 \leq 0$ because $\mathcal{E} \subset \tilde{L}_E \in |\tilde{L}|$. If $\mathcal{E}^2 = 0$ then $\mathcal{E}.G_s = 2$ and $\tilde{L}_E = a\mathcal{E}$ for some $a \geq 1$. Hence $a = 4$ since $\tilde{L}.G_s = 8$. A contradiction, since $|\tilde{L}|$ does not have multiple curves.

So $\deg(z) = 0$ and $\mathcal{E}^2 = -1$. By the definition of canonical resolution $L_E := \omega(\mathcal{E})$ can not be a point, therefore L_E is a line passing through γ and $L_E \neq L_j, j = 1, \dots, n$. Since $L_E^2 - \mathcal{E}^2 = 2$ there is exactly one point $y_i \neq \gamma$ lying on L_E . Analogously to Lemma 4.2 we get that y_i is an 8-tuple point of G . A contradiction.

If $\tilde{\rho}|_{\bar{E}}$ is a double cover. Then $\bar{E}^2 = 2\mathcal{E}^2 < 0$ is even and hence $\bar{E}^2 \leq -2$. Therefore, $\bar{E}.(E_1 + \dots + E_{2n+\delta_1}) = -1 - \bar{E}^2$ is an odd (non zero) number and it is equal to the number of E_i 's which meet \bar{E} since E is smooth. By the Hurwitz formula we have $\bar{E}.(E_1 + \dots + E_{2n+\delta_1}) \leq \bar{E}.R^* = \mathcal{E}.G_s = 2$ which yields $\bar{E}^2 = -2$ and $\mathcal{E}^2 = -1$. The usual calculation shows that $\omega(\mathcal{E})$ can not be neither a point nor a line through γ . Whence $\hat{S} = S$ and the claim follows. \square

Proposition 4.4. *Let S be the smooth minimal model of a Du Val double plane of type \mathcal{D}_n with $n \geq 2$.*

Then $q(S) = 0$ unless $q(S) = p_g(S) = 1$. More precisely, let \mathcal{P} be the set of $n + \delta_1 + \delta_2$ points $\{p_1, \dots, p_n, q_1, \dots, q_{\delta_1}, r_1, \dots, r_{\delta_2}\}$. Then $p_g(S) = q(S) = 1$ if and only if $n + \delta_1 + \delta_2 = 6$ and

- *either no point of \mathcal{P} is infinitely near to γ and the points of \mathcal{P} lie on a conic;*
- *or exactly a point $p \in \mathcal{P}$ is infinitely near to γ and there is a conic passing through the set of points $\{\gamma\} \cup \mathcal{P} \setminus \{p\}$.*

Proof. Recall that

$$\begin{aligned} p_g(S) &= p_g(S^*) = h^0(W_s, \mathcal{O}_{W_s}(K_{W_s} + \Delta_s)) + h^0(W_s, \mathcal{O}_{W_s}(K_{W_s})) = \\ &= h^0(W_s, \mathcal{O}_{W_s}(K_{W_s} + \Delta_s)) \end{aligned}$$

where $\Delta_s \in \text{Pic}(W_s)$ is such that $G_s \equiv 2\Delta_s$.

Suppose that $q(S^*) \neq 0$, hence $p_g(S^*) \neq 0$ and there exists a curve $C \in |K_{W_s} + \Delta_s|$. We have

$$\begin{aligned} C &\equiv \omega^*(K_{\mathbb{P}^2} + \Delta) - \sum \left(\left\lfloor \frac{m_i}{2} \right\rfloor - 1 \right) \mathcal{E}_i^* \equiv \\ &\equiv \omega^*((2+n)l) - n\mathcal{E}_1^* - \sum_{i=2}^{n+1} (\mathcal{E}_i^* + 2\mathcal{E}_{n+i}^*) - \sum_{i=2n+2}^{2n+\delta_2+1} \mathcal{E}_i^* - \sum_{i=2n+2+\delta_2+1}^{2n+\delta_2+2\delta_2+1} \mathcal{E}_i^* \end{aligned}$$

where l is a line in \mathbb{P}^2 .

On the other hand, using the notations introduced before, we get the following equalities

$$\begin{aligned} C.C_i &= C.(\omega^*(l) - \mathcal{E}_1^* - \mathcal{E}_{i+1}^* - \mathcal{E}_{i+n+1}^*) = -1, \quad i = 1, \dots, n; \\ C.C_{i+n} &= C.(\mathcal{E}_{i+1}^* - \mathcal{E}_{i+n+1}^*) = -1, \quad i = 1, \dots, n; \\ C.C_{i+2n} &= C.(\mathcal{E}_{i+2n+\delta_2}^* - \mathcal{E}_{1+2n+\delta_2+\delta_1}^*) = -1, \quad i = 1, \dots, \delta_1 \end{aligned}$$

which imply that the C_i 's are fixed components of $|C|$. Therefore, we can write

$$|C| = |\omega^*(2l) - \sum_{i=2}^{n+1} \mathcal{E}_{n+i}^* - \sum_{i=2n+2}^{2n+\delta_2+1} \mathcal{E}_i^* - \sum_{i=2n+2+\delta_2}^{2n+\delta_2+\delta_2+1} \mathcal{E}_i^* + \sum_{i=1}^{2n+\delta_1} C_i|$$

and so

$$p_g(S^*) = h^0(W_s, \mathcal{O}_{W_s}(\mathcal{C}))$$

where

$$\mathcal{C} \in |\omega^*(2l) - \sum_{i=1}^n \omega^*(p_i) - \sum_{i=1}^{\delta_1} \omega^*(q_i) - \sum_{i=1}^{\delta_2} \omega^*(r_i)|$$

Now there are two cases to be considered: either at least a point of the set $\mathcal{P} = \{p_1, \dots, p_n, q_1, \dots, q_{\delta_1}, r_1, \dots, r_{\delta_2}\}$ is infinitely near to $\gamma = y_0$, or not.

First we consider the second case. Then $p_g(S^*)$ is equal to the dimension of the vector space $V_m \subset H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2l))$ consisting of those conics in \mathbb{P}^2 which pass through the $m = n + \delta_1 + \delta_2 \leq 6$ points of \mathcal{P} .

It is well known that the dimension of V_m is greater than or equal to $6 - m$ and by Proposition 4.3 $q(S^*) = 0$ if and only if the equality holds. In particular, we can assume $m > 3$.

If $4 \leq m \leq 5$ then $p_g(S^*) > 6 - m$ if and only if there exists a line passing through at least 4 points of \mathcal{P} . Whereas if $m = 6$ then $p_g(S^*) > 0$ if and only if all the points lie on a conic and $p_g(S^*) > 1$ if and only if at least 5 points are contained in a line.

Assume that $n + \delta_1 + \delta_2 = 4$ and suppose that there exists a line L' passing through the four points. If $L' \not\subset G$ we get $10 + 2n = L'.G \geq 5n + 3(4 - n) = 2n + 12$, a contradiction. On the other hand if $L' \subset G$ then L' is tangent to G at each $[5, 5]$ -point (resp. $[3, 3]$ -point) and hence $10 + 2n - 1 = (G - L').L' = 8n + 3(4 - n) = 5n + 12$, a contradiction.

Now assume that $n + \delta_1 + \delta_2 = 5$ and suppose that there exists a line L' passing through four points of \mathcal{P} . In particular, there is at most one $[5, 5]$ -point which does not lie on L' . Hence, either $10 + 2n = G.L' \geq 5(n - 1) + 1 + 3(4 - (n - 1)) = 2n + 11$ or $9 + 2n = (G - L').L' \geq 8(n - 1) + 3(4 - (n - 1)) = 5n + 7$ depending on $L' \not\subset G$ or $L' \subset G$. A contradiction.

Finally, assume that $n + \delta_1 + \delta_2 = 6$ and suppose that a line L' passes through 5 of the points. Then there is at most one $[5, 5]$ -point which does not lie on

L' . Hence, either $10 + 2n = G.L' \geq 5(n-1) + 1 + 3(5 - (n-1)) = 2n + 14$ or $9 + 2n = (G - L').L' \geq 8(n-1) + 3(5 - (n-1)) = 5n + 10$ depending on $L' \not\subset G$ or $L' \subset G$. A contradiction.

Therefore, $p_g(S^*) > 6 - n - \delta_1 - \delta_2$ implies that $n + \delta_1 + \delta_2 = 6$, the points of \mathcal{P} lie on a conic and no five of them are collinear. Hence $\chi(S) = 1$ and $p_g(S^*) = q(S^*) = 1$.

Next we discuss the other case. Let us denote by $\overline{\mathcal{E}}_1 \subset W_s$ the strict transform of \mathcal{E}_1 . First suppose that exactly a point, say $p \in \mathcal{P}$, is infinitely near to $y_0 = \gamma$. Hence

$$\mathcal{C}.\overline{\mathcal{E}}_1 \leq \mathcal{C}(\mathcal{E}_1^* - \sum_{i \geq 2} c_i \mathcal{E}_i^* \leq \mathcal{C}(\mathcal{E}_1^* - \mathcal{E}^*) = -1$$

where $\mathcal{E}^* = \omega^*(p)$ and c_i is equal to 1 or 0 depending on y_{i-1} is infinitely near to y_0 or not. Therefore, $\overline{\mathcal{E}}_1$ is a fixed component of $|\mathcal{C}|$ and $p_g(S^*)$ is easily seen to be equal to the dimension of the vector space consisting of those conics passing through the set of points $\{\gamma\} \cup \mathcal{P} \setminus \{p\}$. As before we get the claim.

Now suppose that at least two points of \mathcal{P} are infinitely near to y_0 and let p, q be two of them. Let $\overline{L}_i \subset W_1$ be the strict transform of L_i under ω_1 , $i = 1, \dots, n$, and denote by $\text{mult}_p(G_1 - \sum \overline{L}_i)$, $\text{mult}_q(G_1 - \sum \overline{L}_i)$ the multiplicity of $G_1 - \sum L_i$ at p and q , respectively. Then from the inequalities

$$6 \leq \text{mult}_p(G_1 - \sum L_i) + \text{mult}_q(G_1 - \sum L_i) \leq (G_1 - \sum \overline{L}_i).\mathcal{E}_1 = 2 + n$$

it follows that $n \geq 4$ and it is easy to check that one has $n + \delta_1 + \delta_2 = 6$, where $n \in \{4, 5, 6\}$, and that there are exactly two points infinitely near to γ which have to be respectively p_1, p_2 ; q_1, q_2 .

We consider the case $n = 6$, the others are completely analogous. Then we have $\mathcal{C}.\overline{\mathcal{E}}_1 = \mathcal{C}(\mathcal{E}_1^* - \mathcal{E}_2^* - \mathcal{E}_3^*) = -2$ and $|\mathcal{C}| = |\mathcal{C}'| + \overline{\mathcal{E}}_1$ where \mathcal{C}' is strict transform of a conic through γ, p_3, \dots, p_6 . Hence $1 \leq p_g(S^*) = q(S^*) \leq 2$ and $p_s(S^*) = 2$ if and only if p_3, \dots, p_6 lie on a line.

Suppose that $p_g(S^*) = 2$. Let L' be the line passing through p_3, \dots, p_6 and consider the linear system $|\omega^*(4l) - 2\mathcal{E}_1^* - \sum_{i=1}^6 (\mathcal{E}_{i+1}^* + \mathcal{E}_{i+n+1}^*)|$. Let $\overline{F}_1, \overline{F}_2$ be the strict transforms on W_s of $F_1 := L_1 + L_2 + 2L'$ and $F_2 := L_3 + \dots + L_6$, respectively. Then, $\overline{F}_1, \overline{F}_2$ do not have common components and $\overline{F}_j \in |\omega^*(4l) - 2\mathcal{E}_1^* - \sum_{i=1}^6 (\mathcal{E}_{i+1}^* + \mathcal{E}_{i+n+1}^*)|$, $j = 1, 2$. Arguing as in Lemma 2.10 we get that the general element $F \in |\omega^*(4l) - 2\mathcal{E}_1^* - \sum_{i=1}^6 (\mathcal{E}_{i+1}^* + \mathcal{E}_{i+n+1}^*)|$ is a smooth curve of genus 2 such that $F.G_s = 0$. A contradiction, because we are assuming that S does not present the standard case. \square

Remark 4.5. If $n = 1$ and the 4-tuple point lying on L_1 is not infinitely near to the $[5, 5]$ -point, the above theorems holds also for S^* of type \mathcal{D}_1 .

Case B). X' is of type \mathcal{D}_n , $n < 2$.

Proposition 4.6. *Let S be the minimal model a Du Val double plane of type \mathcal{D}_n with $n \leq 1$. Then*

a)

$$\begin{aligned} K_S^2 &= 8 - n - \delta_1 - 2\delta_2 \\ p_g(S) - q(S) &= 6 - n - \delta_1 - \delta_2 \\ q(S) &= 0 \text{ unless } p_g(S) = q(S) = 1 \text{ and } K_S^2 = 3; \end{aligned}$$

- in particular, $p_g(S) = q(S) = 1$ if and only if $n = 1, \delta_1 = 5, \delta_2 = 0$ and the points p_1, q_1, \dots, q_5 lie on a conic;
- b) the bicanonical map of S factors through ρ ;
- c) either $p_g(S) = 6, K_S^2 = 8$ and K_S is ample or there is a rational pencil $|H|$ on S such that:
- i) the general member $H \in |H|$ is a smooth hyperelliptic curve of genus 3;
 - ii) the bicanonical map of S induces the hyperelliptic involution on the general $H \in |H|$.
 - iii) either $p_g(S) = 6, K_S^2 = 8$ (K_S is not ample) and $|H|$ does not have base points or $|H|$ has one base point.

Proof. If $n = \delta_1 = 0$ (and then $\delta_2 = 0$) and G is smooth, then $S = S^*$ and it is easily seen that (a), (b) hold. In particular, $p_g = 6, K_S^2 = 8$ and K_S is ample.

Hence, we can assume that

- $y_0 = p$ if $n = \delta_1 = 0$;
- $y_0 = q_1$ if $n = 0, \delta_1 \geq 1$;
- $y_0 = p_1$ if $n = 1$.

where in the first case $p \in \mathbb{P}^2$ is a (non essential) singular point of G .

Now we proceed as in the proof of Proposition 4.3. Let \tilde{H} be the pull back to S^* of a general line passing through y_0 . Hence, $|\tilde{H}|$ is a pencil of (smooth) hyperelliptic curves of genus 3. In particular,

$$\tilde{H}^2 = 0, \quad \tilde{H}.K_{S^*} = 4, \quad R^*.\tilde{H} = 8.$$

Let $\pi_1 : S^* \rightarrow S'$ be the birational morphism which contracts the (-1) -curves $E_1, \dots, E_{\delta_1+2n}$ arising from the $[r, r]$ -points, $r = 3, 5$, and from L_1 (resp. $\pi_1 = id$ if $\delta_1 + n = 0$). We set $H' = \pi_1(\tilde{H})$.

Note that if $\delta_1 + n > 0$ we can assume $E_1 = \tilde{\rho}^{-1}(\bar{\mathcal{E}}_1)$ where $\bar{\mathcal{E}}_1 \subset G_s$ is the strict transform of $\mathcal{E}_1 = \omega_1^{-1}(y_0)$. Hence, $E_i.\tilde{H} = 0$ for each $i > 1$ and

$$H'^2 = E_1.\tilde{H} = \begin{cases} 1 & \text{if } n + \delta_1 > 0 \\ 0 & \text{if } n + \delta_1 = 0 \end{cases}$$

By Lemma 4.1 $p_g(S') - q(S') = 6 - \delta_1 - \delta_2 - n$ and we get the following table for $(\chi(S') - 1, K_{S'}^2)$:

$\chi(S) - 1$	K_S^2
6	8
5	7
4	6
3	5
2	4
1	3
0	2

As before it suffices to show that $S' = S$. In particular, recall that as S does not present the standard case, if $q(S) > 0$ then $K_S^2 > 2\chi(S)$ (cfr. [17]).

Let us suppose that $E \subset S'$ is a (-1) -curve and define $H = \pi_2(H')$, where $\pi_2 \circ \pi_1 = \pi$. First we prove that $E.H' = 0$.

Suppose that $E.H' > 0$.

If $n + \delta_1 > 0$ the Hodge Index Theorem gives $K_S^2 \leq 2$. A contradiction, since $K_S^2 \geq K_{S'}^2 + 1 \geq 3$.

If $n + \delta_1 = 0$, then $S' = S^*$ and $K_{S'}^2 = 8$. Since $H^2 \geq 1$ and $H.K_S \leq 3$ it follows from the Hodge Index Theorem that $K_S^2 = 9$ and K_S is numerically equivalent to $3H$. In particular $\pi = \pi_1$ contracts exactly E and $E.\tilde{H} = 1$.

If $E \not\subset R^*$, then the involution σ induced on S has no isolated fixed points and by Proposition 1.2 we get $0 = R.K_S - 20$. A contradiction, since we have that $R.K_S = 3R.H \geq 3R^*.\tilde{H} = 24$.

Whence $E \subset R^*$ and $\mathcal{E} := \tilde{\rho}(E)$ is a (-2) -curve belonging to G_s . By Lemma 4.2 $\mathcal{E}' := (\omega_2 \circ \dots \circ \omega_s)(\mathcal{E})$ is a curve on W_1 . Denote by $\bar{L} := \omega_1^*(L) - \mathcal{E}_1$ the strict transform on W_1 of a general line passing through y_0 . Since $S^* \rightarrow W_1$ is a (finite) double cover in a neighborhood of \bar{L} , we have $\bar{L}.\mathcal{E}' = \tilde{H}.E = 1$ and so \mathcal{E}' is smooth. Therefore, $\omega_1(\mathcal{E}') \subset \mathbb{P}^2$ is a reduced and irreducible curve of degree $d \geq 1$ with multiplicity $d - 1 = \mathcal{E}'.\mathcal{E}_1$ at y_0 and smooth elsewhere.

It follows that $\mathcal{E}'^2 = 2d - 1 \geq 1$ and we can assume $y_1 \in \mathcal{E}'$. On the other hand G_1 does not have essential singularities and hence $(\omega_2 \circ \dots \circ \omega_t \circ \tilde{\rho})^*(y_1)$ is a (-2) -cycle on S^* (cfr. [1]). In particular, there is a (-2) -curve $E' \subset S^*$ such that $E.E' = 1$ and hence $\pi(E') \subset S$ is a (-1) -curve. A contradiction.

Therefore, $E.H' = 0$. Now arguing as in the proof of Proposition 4.3 we get a contradiction and then $S' = S$. \square

Proposition 4.7. *Let S^* be the canonical resolution of a Du Val double plane of type \mathcal{D}_0 such that $\delta_1 \geq 2$. Let l_i be the line tangent to G at $q_i, i = 1, \dots, \delta_1$. Then $l_j \neq l_k$ for some $j \neq k$ if and only if S^* is the canonical resolution of a Du Val double plane of type \mathcal{D}_1 with a 4-tuple point and $\delta_1 - 2$ $[3, 3]$ -points.*

Proof. Assume that S^* is the canonical resolution of a Du Val double plane of type \mathcal{D}_0 and suppose that $[q'_1 \rightarrow q_1], [q'_2 \rightarrow q_2]$ are such that $l_1 \neq l_2$. Then we can perform the quadratic transformation of the plane $\lambda_{q_1, q'_1, q_2} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ centered at q_1, q'_1, q_2 . Let G' be the proper transform of G under λ_{q_1, q'_1, q_2} and let L_1 be the image of the exceptional curve arising from q_1 . Then it is easily seen that G' is a reduced curve of degree 11 with a triple point and a $[4, 4]$ -point lying on the line L_1 , a 4-tuple point at the image of q'_1 and $\delta_1 - 2$ $[3, 3]$ -points at the image of q_3, \dots, q_{δ_1} .

Therefore, we have the following commutative diagram

$$\begin{array}{ccc}
 & S^* & \\
 \omega \swarrow & & \searrow \omega' \\
 G \subset \mathbb{P}^2 & \xrightarrow{\lambda_{q_1, q'_1, q_2}} & \mathbb{P}^2 \supset G' + L_1
 \end{array}$$

where ω' is a morphism because we blow up singular points of G . Now arguing as in Lemma 1.3 one sees that S^* is the canonical resolution of the double cover of \mathbb{P}^2 branched along $G' + L_1$.

For the converse, perform the quadratic transformation of \mathbb{P}^2 centered at p_1, p'_1, r_1 . \square

Case C). X' is of type \mathcal{B} or \mathcal{D} . This is the easiest case. In fact, arguing as above one gets the following:

Proposition 4.8. *Let S be the minimal model a Du Val double plane of type \mathcal{B} or \mathcal{D} . Then $q(S) = 0$ and the bicanonical map of S factors through ρ . Moreover,*

- a) *if S is of type \mathcal{D} , then $p_g(S) = 3, K_S^2 = 2$ and K_S is ample;*
- b) *if S is of type \mathcal{B} , then $p_g(S) = 6, K_S^2 = 9$ and there is a rational pencil $|H|$ on S such that:*
 - i) *the general member $H \in |H|$ is a smooth hyperelliptic curve of genus 3;*
 - ii) *the bicanonical map of S induces the hyperelliptic involution on the general $H \in |H|$.*
 - iii) *$|H|$ has one base point .*

5. CONCLUSION AND REMARKS

We collect some corollaries of the main theorem. Throughout the end S will be a minimal surface of general type not presenting the *standard case*.

Corollary 5.1. *Let S be the smooth minimal model of a Du Val double plane of type \mathcal{D}_n . If $n \geq 2$ then $(\mathbb{Z}_2)^{n-1} \subseteq \text{Tors}(S)$.*

Proof. Because the rational pencil $|H|$ has n pairwise distinct double curves the claim is clear. \square

As we remarked in the introduction, if S is the smooth minimal model of a double plane with $p_g(S) \geq 2$ then the bicanonical map of S has degree 2, because S is regular. In the following corollary we show that if $p_g(S) \leq 1$ then φ_{2K} may have degree greater than 2.

Corollary 5.2. *Let S be the smooth minimal model of a Du Val double plane with $p_g(S) = 1$, $q(S) = 0$ and $K_S^2 = 2$. Then φ_{2K} has degree 4 and S_2 is a quadric cone in \mathbb{P}^3 .*

Proof. By Proposition 4.3, Proposition 4.4 and Proposition 4.6, S^* is of type \mathcal{D}_2 and the branch curve G has $\delta_1 = 0$ $[3, 3]$ -points and $\delta_2 = 3$ 4-tuple points.

Although the claim follows from a result of Catanese, Debarre (cfr. [4], Proposition 1.5) and the previous Corollary, it can also be proved with the same argument we used before.

As the bicanonical map of S factors through the involution induced by the double cover, we have the commutative diagram

$$\begin{array}{ccc} S^* & \xrightarrow{\pi} & S \\ \bar{\rho} \downarrow & & \downarrow \varphi_{2K} \\ \mathbb{P}^2 & \xleftarrow{\omega} \hat{W}_s \xrightarrow{\varphi_F} & S_2 \end{array}$$

where φ_F is the morphism defined by the linear system

$$\begin{aligned} |F| &= |2K_{W_s} + G_s - \sum_1^4 C_i| = \\ &= |\omega^*(6l) - \omega^*(2\gamma) - \sum_{i=1}^2 \omega^*(2p_i + 2p'_i) - \sum_{i=1}^3 \omega^*(2r_i)| \end{aligned}$$

(here l is a line in \mathbb{P}^2). Let \mathcal{C}_3 denote the strict transform under ω of a general cubic in \mathbb{P}^2 passing through the set $\mathcal{P} = \{\gamma, p_1, p'_1, p_2, p'_2, r_1, r_2, r_3\}$. Then \mathcal{C}_3 is a smooth curve of genus 1 and the linear system $|\mathcal{C}_3|$ has one base point $p \in \hat{W}$.

Now one sees that $|F|$ cuts a g_2^1 on the general curve in $|\mathcal{C}_3|$ and so φ_F has degree greater than 2. It follows that φ_{2K} has degree $d \geq 4$ and S_2 is a surface of degree $\frac{8}{d}$ in \mathbb{P}^3 . Whence, $d = 4$ and $\varphi_{2K}(S) = \varphi_F(W_s)$ is a quadric cone with vertex $\varphi_F(p)$ and ruling $\varphi_F(\mathcal{C}_3)$. \square

Proposition 5.3. *Let S be a smooth minimal surface of general type with $p_g(S) = 0$. If the bicanonical map has degree 2 and S does not present the standard case, then*

- 1) *either $K_S^2 = 3$ and $S_2 \subset \mathbb{P}^2$ is an Enriques surface,*
- 2) *or S is the smooth minimal model of a Du Val double plane of type \mathcal{D}_n with K_S^2 and n as in the following table:*

K_S^2	2	3	4	5	6	7	8
n	0,1,2,3	1,2,3	2,3,4	3,4	4,5	5	6

Moreover, in case 2) there is a rational pencil $|H|$ whose general member is a smooth hyperelliptic curve of genus 3 such that

- the bicanonical map of S induces the hyperelliptic involution on the general curve $H \in |H|$;
- if $n \leq 1$ then $|H|$ has one base point;
- if $n \geq 2$ then $|H|$ is base points free and has n double fibres.

Proof. By [18] if S_2 is not rational then $K_S^2 = 3, 4$ and S_2 is an Enriques surface. In [11] M.Mendes Lopes and R.Pardini show that the case $K_S^2 = 4$ does not occur.

Now the claim follows by Theorem 0.1, Proposition 4.3, Proposition 4.6 and Proposition 4.8. \square

Remark 5.4. We remark that the above result was partially proved by R.Pardini and M.Mendes Lopes. In fact, they classify surfaces of general type with $6 \leq p_g \leq 8$ and bicanonical map of degree two in [12],[13],[14] where they also construct examples of such surfaces.

As we remarked in the introduction, we get an analogous result for regular surfaces with $p_g = 1$.

Proposition 5.5. *Let S be a smooth minimal surface of general type with $q(S) = 0$ and $p_g(S) = 1$. If the bicanonical map has degree 2, S does not present the standard case and the bicanonical image S_2 is not a K3 surface, then S is the smooth minimal model of a Du Val double plane of type \mathcal{D}_n with K_S^2 and n as in the following table:*

K_S^2	2	3	4	5	6	7	8
n	2	0,1,2	1,2,3	2,3	3,4	4	5

Moreover, there is a rational pencil $|H|$ whose general member is a smooth hyperelliptic curve of genus 3 such that

- the bicanonical map induces the hyperelliptic involution on the general $H \in |H|$;
- if $n \leq 1$ then $|H|$ has one base point;
- if $n \geq 2$ then $|H|$ is base points free and has n double fibres.

Finally, we get a partial result concerning the case $p_g(S) = q(S) = 1$.

Proposition 5.6. *Let S be a smooth minimal surface of general type with $p_g(S) = q(S) = 1$ and $7 \leq K_S^2 \leq 8$. Assume that the bicanonical map of S has degree 2 and that S does not present the standard case. Then*

- If $K_S^2 = 7$, then S is the smooth minimal model of a double plane branched along a reduced curve $G = G' + L_1 + \dots + L_5$, where G' has degree 15 and L_1, \dots, L_5 are lines meeting at a point γ . The essential singularities of G are a 12-tuple point at γ , a $[5, 5]$ -point $[p'_i \rightarrow p_i]$ on $L_i, i = 1, \dots, 5$, a $[3, 3]$ -point $[q'_1 \rightarrow q_1]$. The points p_1, \dots, p_5, q_1 lie on a conic.
- If $K_S^2 = 8$, then S is the smooth minimal model of a double plane branched along a reduced curve $G = G' + L_1 + \dots + L_6$, where G' has degree 16 and L_1, \dots, L_6 are lines meeting at a point γ . The essential singularities of G are a 14-tuple point at γ , a $[5, 5]$ -point $[p'_i \rightarrow p_i]$ on $L_i, i = 1, \dots, 6$. The points p_1, \dots, p_6 lie on a conic.

Proof. By [18], Theorem 3, the bicanonical image S_2 is a rational surface. Therefore, we can apply Theorem 0.1 and then the results of Section 4. \square

Remark. Surfaces with $p_g = q = 1, K^2 = 8$ and bicanonical map of degree 2 are studied in detail and classified by F.Polizzi in his PhD thesis (cfr.[15]). In particular, he constructs such surfaces as the quotient of the product of two curves by a finite group.

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